

On Schrödinger systems with cubic dissipative nonlinearities of derivative type

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Abstract: Consider the initial value problem for systems of cubic derivative nonlinear Schrödinger equations in one space dimension with the masses satisfying a suitable resonance relation. We give structural conditions on the nonlinearity under which the small data solution gains an additional logarithmic decay as $t \rightarrow +\infty$ compared with the corresponding free evolution.

Key Words: Derivative nonlinear Schrödinger systems; Nonlinear dissipation; Logarithmic time-decay.

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1 Introduction

Consider the initial value problem for the system of nonlinear Schrödinger equations of the following type:

$$\begin{cases} \mathcal{L}_{m_j} u_j = F_j(u, \partial_x u), & t > 0, x \in \mathbb{R}, j = 1, \dots, N, \\ u_j(0, x) = \varphi_j(x), & x \in \mathbb{R}, j = 1, \dots, N, \end{cases} \quad (1.1)$$

where $\mathcal{L}_{m_j} = i\partial_t + \frac{1}{2m_j}\partial_x^2$, $i = \sqrt{-1}$, $m_j \in \mathbb{R} \setminus \{0\}$, and $u = (u_j(t, x))_{1 \leq j \leq N}$ is a \mathbb{C}^N -valued unknown function. The nonlinear term $F = (F_j)_{1 \leq j \leq N}$ is always assumed to be a cubic homogeneous polynomial in $(u, \partial_x u, \bar{u}, \overline{\partial_x u})$. Our main interest is how the combinations of $(m_j)_{1 \leq j \leq N}$ and the structures of $(F_j)_{1 \leq j \leq N}$ affect large-time behavior of the solution u to (1.1). Before going into details, let us first recall some known results briefly and clarify our motivation.

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One of the most typical nonlinear Schrödinger equations appearing in various physical settings is

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda|u|^2 u, \quad t > 0, \ x \in \mathbb{R} \quad (1.2)$$

with $\lambda \in \mathbb{R}$. What is interesting in (1.2) is that the large-time behavior of the solution is actually affected by the nonlinearity even if the initial data is sufficiently small, smooth and decaying fast as $|x| \rightarrow \infty$. To be more precise, it is shown in [1] that the solution to (1.2) with small initial data behaves like

$$u(t, x) = \frac{1}{\sqrt{it}} \alpha(x/t) e^{i\{\frac{x^2}{2t} - \lambda|\alpha(x/t)|^2 \log t\}} + o(t^{-1/2}) \quad \text{as } t \rightarrow \infty$$

with a suitable \mathbb{C} -valued function $\alpha(y)$. An important consequence of this asymptotic expression is that the solution decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R}_x)$, while it does not behave like the free solution unless $\lambda = 0$. In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. If $\lambda \in \mathbb{C}$, another kind of long-range effect can be observed. Indeed, it is verified in [15] that the small data solution to (1.2) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \rightarrow \infty$ if $\text{Im } \lambda < 0$ (see also [17]). This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect. Among several extensions of this result (see e.g., [3], [9], [11], [12], [13] etc. and the references cited therein), let us focus on the following two cases: (i) the case where the nonlinearity depends also on $\partial_x u$, and (ii) the case of systems.

(i) Let us consider the single nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = G(u, \partial_x u), \quad t > 0, \ x \in \mathbb{R}, \quad (1.3)$$

where G is a cubic homogeneous polynomial in $(u, \partial_x u, \bar{u}, \overline{\partial_x u})$ with complex coefficients, and satisfies the gauge invariance

$$G(e^{i\theta} v, e^{i\theta} w) = e^{i\theta} G(v, w), \quad \theta \in \mathbb{R}, \ (v, w) \in \mathbb{C} \times \mathbb{C}. \quad (1.4)$$

According to [3], the solution to (1.3) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \rightarrow \infty$ if

$$\sup_{\xi \in \mathbb{R}} \text{Im } G(1, i\xi) < 0. \quad (1.5)$$

However, the approach of [3] does not work well in the case of systems, because this additional logarithmic decay result is a consequence of the explicit asymptotic profile of the solution $u(t, x)$, which becomes no longer simple in the coupled case.

- (ii) For nonlinear Schrödinger systems, an additional logarithmic decay result is first obtained by [5]. Strictly saying, two-dimensional quadratic nonlinear Schrödinger systems are treated in [5], but we can adopt the method of [5] directly to one-dimensional cubic nonlinear Schrödinger systems, as pointed in [9]. When we restrict ourselves to a two-component model

$$\begin{cases} \mathcal{L}_{m_1} u_1 = \lambda_1 |u_1|^2 u_1 + \nu_1 \overline{u_1}^2 u_2, \\ \mathcal{L}_{m_2} u_2 = \lambda_2 |u_2|^2 u_2 + \nu_2 u_1^3, \end{cases} \quad t > 0, x \in \mathbb{R} \quad (1.6)$$

with $\lambda_1, \lambda_2, \nu_1, \nu_2 \in \mathbb{C}$ and $m_1, m_2 \in \mathbb{R} \setminus \{0\}$, then the result of [5] can be read as follows: the solution to (1.6) decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \rightarrow \infty$ if

$$m_2 = 3m_1, \quad (1.7)$$

$$\operatorname{Im} \lambda_j < 0, \quad j = 1, 2, \quad (1.8)$$

and

$$\kappa_1 \nu_1 = \kappa_2 \overline{\nu_2} \quad \text{with some } \kappa_1, \kappa_2 > 0 \quad (1.9)$$

(see Example 2.1 in [9] for the detail). The advantage of the method of [5] is that it does not rely on the explicit asymptotic profile at all. However, it is not straightforward to apply this approach in the derivative nonlinear case, because we need suitable pointwise a priori estimates not only for the solution itself but also for its derivatives without breaking good structure in order to apply the method of [5].

The purpose of this paper is to unify (i) and (ii). More precisely, we will introduce structural conditions on $(F_j)_{1 \leq j \leq N}$ and $(m_j)_{1 \leq j \leq N}$ under which the small data solution to the derivative nonlinear Schrödinger system (1.1) gains an additional logarithmic decay as $t \rightarrow +\infty$ compared with the corresponding free evolution.

2 Main Results

In the subsequent sections, we will use the following notations: We set $I_N = \{1, \dots, N\}$ and $I_N^\sharp = \{1, \dots, N, N+1, \dots, 2N\}$. For $z = (z_j)_{j \in I_N} \in \mathbb{C}^N$, we write

$$z^\sharp = (z_k^\sharp)_{k \in I_N^\sharp} := (z_1, \dots, z_N, \overline{z_1}, \dots, \overline{z_N}) \in \mathbb{C}^{2N}.$$

Then general cubic nonlinear term $F = (F_j)_{j \in I_N}$ can be written as

$$F_j(u, \partial_x u) = \sum_{l_1, l_2, l_3=0}^1 \sum_{k_1, k_2, k_3 \in I_N^\sharp} C_{j, k_1, k_2, k_3}^{l_1, l_2, l_3} (\partial_x^{l_1} u_{k_1}^\sharp) (\partial_x^{l_2} u_{k_2}^\sharp) (\partial_x^{l_3} u_{k_3}^\sharp)$$

with suitable $C_{j,k_1,k_2,k_3}^{l_1,l_2,l_3} \in \mathbb{C}$. With this expression of F , we define $p = (p_j(\xi; Y))_{j \in I_N} : \mathbb{R} \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ by

$$p_j(\xi; Y) := \sum_{l_1, l_2, l_3=0}^1 \sum_{k_1, k_2, k_3 \in I_N^\#} C_{j,k_1,k_2,k_3}^{l_1,l_2,l_3} (i\tilde{m}_{k_1}\xi)^{l_1} (i\tilde{m}_{k_2}\xi)^{l_2} (i\tilde{m}_{k_3}\xi)^{l_3} Y_{k_1}^\# Y_{k_2}^\# Y_{k_3}^\#$$

for $\xi \in \mathbb{R}$ and $Y = (Y_j)_{j \in I_N} \in \mathbb{C}^N$, where

$$\tilde{m}_k = \begin{cases} m_k & (k = 1, \dots, N), \\ -m_{(k-N)} & (k = N+1, \dots, 2N). \end{cases}$$

In what follows, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$ the standard scalar product in \mathbb{C}^N , i.e.,

$$\langle z, w \rangle_{\mathbb{C}^N} = \sum_{j=1}^N z_j \overline{w_j}$$

for $z = (z_j)_{j \in I_N}$ and $w = (w_j)_{j \in I_N} \in \mathbb{C}^N$.

Now let us introduce the following conditions:

(a) For all $j \in I_N$ and $k_1, k_2, k_3 \in I_N^\#$,

$$m_j \neq \tilde{m}_{k_1} + \tilde{m}_{k_2} + \tilde{m}_{k_3} \text{ implies } C_{j,k_1,k_2,k_3}^{l_1,l_2,l_3} = 0, \quad l_1, l_2, l_3 \in \{0, 1\}.$$

(b₀) There exists an $N \times N$ positive Hermitian matrix A such that

$$\operatorname{Im} \langle p(\xi; Y), AY \rangle_{\mathbb{C}^N} \leq 0$$

for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^N$.

(b₁) There exist an $N \times N$ positive Hermitian matrix A and a positive constant C_* such that

$$\operatorname{Im} \langle p(\xi; Y), AY \rangle_{\mathbb{C}^N} \leq -C_* |Y|^4$$

for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^N$.

(b₂) There exist an $N \times N$ positive Hermitian matrix A and a positive constant C_{**} such that

$$\operatorname{Im} \langle p(\xi; Y), AY \rangle_{\mathbb{C}^N} \leq -C_{**} \langle \xi \rangle^2 |Y|^4$$

for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^N$, where $\langle \xi \rangle = \sqrt{1 + \xi^2}$.

(b₃) $p(\xi; Y) = 0$ for all $(\xi, Y) \in \mathbb{R} \times \mathbb{C}^N$.

To state the main results, we introduce some function spaces. For $s, \sigma \in \mathbb{Z}_{\geq 0}$, we denote by H^s the L^2 -based Sobolev space of order s , and the weighted Sobolev space $H^{s,\sigma}$ is defined by $\{\phi \in L^2 \mid \langle x \rangle^\sigma \phi \in H^s\}$, equipped with the norm $\|\phi\|_{H^{s,\sigma}} = \|\langle x \rangle^\sigma \phi\|_{H^s}$. The main results are as follows:

Theorem 2.1. Assume the conditions (a) and (b₀) are satisfied. Let $\varphi = (\varphi_j)_{j \in I_N} \in H^3 \cap H^{2,1}$, and assume $\varepsilon := \|\varphi\|_{H^3} + \|\varphi\|_{H^{2,1}}$ is sufficiently small. Then (1.1) admits a unique global solution $u = (u_j)_{j \in I_N} \in C([0, \infty); H^3 \cap H^{2,1})$. Moreover we have

$$\|u(t)\|_{L^\infty} \leq \frac{C\varepsilon}{\sqrt{1+t}}, \quad \|u(t)\|_{L^2} \leq C\varepsilon$$

for $t \geq 0$, where C is a positive constant not depending on ε .

Theorem 2.2. Assume the conditions (a) and (b₁) are satisfied. Let u be the global solution to (1.1), whose existence is guaranteed by Theorem 2.1. Then we have

$$\|u(t)\|_{L^\infty} \leq \frac{C\varepsilon}{\sqrt{(1+t)\{1+\varepsilon^2 \log(2+t)\}}}$$

for $t \geq 0$, where C is a positive constant not depending on ε . We also have

$$\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2} = 0.$$

Theorem 2.3. Assume the conditions (a) and (b₂) are satisfied. Let u be as above. Then we have

$$\|u(t)\|_{L^2} \leq \frac{C\varepsilon}{\sqrt{1+\varepsilon^2 \log(2+t)}}$$

for $t \geq 0$, where C is a positive constant not depending on ε .

Theorem 2.4. Assume the conditions (a) and (b₃) are satisfied. Let u be as above. For each $j \in I_N$, there exists $\varphi_j^+ \in L^2(\mathbb{R}_x)$ with $\hat{\varphi}_j^+ \in L^\infty(\mathbb{R}_\xi)$ such that

$$u_j(t) = e^{i\frac{t}{2m_j}\partial_x^2} \varphi_j^+ + O(t^{-1/4+\delta}) \quad \text{in } L^2(\mathbb{R}_x)$$

and

$$u_j(t, x) = \sqrt{\frac{m_j}{it}} \hat{\varphi}_j^+ \left(\frac{m_j x}{t} \right) e^{i\frac{m_j x^2}{2t}} + O(t^{-3/4+\delta}) \quad \text{in } L^\infty(\mathbb{R}_x)$$

as $t \rightarrow +\infty$, where $\delta > 0$ can be taken arbitrarily small, and $\hat{\phi}$ denotes the Fourier transform of ϕ , i.e.,

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} \phi(y) dy.$$

Remark 2.1. In view of the proof of Theorem 2.4 below, we can see that $\varphi^+ = (\varphi_j^+)_{j \in I_N}$ does not identically vanish if the initial data φ is suitably small and does not identically vanish (see Remark 6.1 for the detail). Therefore the solution does not gain an additional logarithmic decay under the conditions (a) and (b₃).

Now let us give several examples which satisfy the above mentioned conditions:

Example 2.1. In the single case (i.e., $N = 1$), we may assume $m_1 = 1$ without loss of generality. Then we can check that the condition (a) is equivalent to the gauge invariance (1.4), and that the condition (1.5) is equivalent to the condition (b₁). Therefore our results above can be viewed as an extension of [3] except the explicit asymptotic profile of the solution. We can also see that our results cover the system (1.6) under the assumptions (1.7), (1.8), (1.9). Indeed, (1.7) plays the role of (a), and (1.8), (1.9) correspond to (b₁) with $A = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$.

Example 2.2. Next let us consider the following two-component system

$$\begin{cases} \mathcal{L}_m u_1 = \lambda_1 |u_1|^2 u_1 + \lambda_2 \overline{u_1} (\partial_x u_1)^2 + i u_2 \partial_x (\overline{u_1}^2), \\ \mathcal{L}_{3m} u_2 = \lambda_3 |u_2|^2 \partial_x u_2 - i(|u_2|^2 + |\partial_x u_2|^2) u_2 - i u_1^2 \partial_x u_1 \end{cases}$$

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $m \in \mathbb{R} \setminus \{0\}$, which is a bit more complicated than (1.6). It is easy to check that the condition (a) is satisfied by this system. Also it follows from simple calculations that

$$\begin{cases} p_1(\xi; Y) = (\lambda_1 - \lambda_2 m^2 \xi^2) |Y_1|^2 Y_1 + 2m \xi \overline{Y_1}^2 Y_2, \\ p_2(\xi; Y) = i(3\lambda_3 m \xi - 1 - 9m^2 \xi^2) |Y_2|^2 Y_2 + 3m \xi Y_1^3. \end{cases}$$

With $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, we have

$$\langle p(\xi; Y), AY \rangle_{\mathbb{C}^2} = 3(\lambda_1 - \lambda_2 m^2 \xi^2) |Y_1|^4 - 2i(1 - 3\lambda_3 m \xi + 9m^2 \xi^2) |Y_2|^4 + 12m \xi \operatorname{Re}(\overline{Y_1}^3 Y_2),$$

whence

$$\operatorname{Im} \langle p(\xi; Y), AY \rangle_{\mathbb{C}^2} = 3(\operatorname{Im} \lambda_1 - \operatorname{Im} \lambda_2 m^2 \xi^2) |Y_1|^4 - \left\{ 2 - \frac{(\operatorname{Re} \lambda_3)^2}{2} + 2 \left(3m \xi - \frac{\operatorname{Re} \lambda_3}{2} \right)^2 \right\} |Y_2|^4.$$

Therefore we see that

- (b₀) is satisfied if $\operatorname{Im} \lambda_1 \leq 0$, $\operatorname{Im} \lambda_2 \geq 0$ and $|\operatorname{Re} \lambda_3| \leq 2$.
- (b₁) is satisfied if $\operatorname{Im} \lambda_1 < 0$, $\operatorname{Im} \lambda_2 \geq 0$ and $|\operatorname{Re} \lambda_3| < 2$.
- (b₂) is satisfied if $\operatorname{Im} \lambda_1 < 0$, $\operatorname{Im} \lambda_2 > 0$ and $|\operatorname{Re} \lambda_3| < 2$.

Example 2.3. Finally we focus on the three-component system

$$\begin{cases} \mathcal{L}_m u_1 = u_2 \partial_x (\overline{u_1} u_2), \\ \mathcal{L}_m u_2 = \overline{u_1} u_2 \partial_x u_3 + 3\overline{u_1} u_3 \partial_x \overline{u_2}, \\ \mathcal{L}_{3m} u_3 = 2u_1^2 \partial_x u_2 - u_2 \partial_x (u_1^2). \end{cases}$$

We can immediately check that this system satisfies (a) and (b₃). Note that this example should be compared with [4], where the null structure in quadratic derivative nonlinear Schrödinger systems in \mathbb{R}^2 is considered in details (see also [7], [8], [16]).

The rest part of this paper is organized as follows: The next section is devoted to preliminaries on basic properties of the operator J_m . In Section 4, we recall the smoothing property of the linear Schrödinger equations. In Section 5, we will get an a priori estimate. After that, The main theorems will be proved in Section 6. The appendix is devoted to the proof of technical lemmas. In what follows, we will denote several positive constants by the same letter C , which is possibly different from line to line.

3 Preliminaries

In this section, we collect several identities and inequalities which are useful for our purpose. We set $J_m = x + i\frac{t}{m}\partial_x$ for non-zero real constant m . Then we can check that $[\partial_x, J_m] = 1$ and $[\mathcal{L}_m, J_m] = 0$, where $[\cdot, \cdot]$ denotes the commutator of two linear operators. We also note that

$$J_m \phi = \frac{it}{m} e^{im\frac{x^2}{2t}} \partial_x (e^{-im\frac{x^2}{2t}} \phi), \quad (3.1)$$

which yields the following useful lemmas.

Lemma 3.1. *Let m, μ_1, μ_2, μ_3 be non-zero real constants satisfying $m = \mu_1 + \mu_2 + \mu_3$. We have*

$$\begin{aligned} J_m(f_1 f_2 f_3) &= \frac{\mu_1}{m} (J_{\mu_1} f_1) f_2 f_3 + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2) f_3 + \frac{\mu_3}{m} f_1 f_2 (J_{\mu_3} f_3), \\ J_m(f_1 f_2 \overline{f_3}) &= \frac{\mu_1}{m} (J_{\mu_1} f_1) f_2 \overline{f_3} + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2) \overline{f_3} + \frac{\mu_3}{m} f_1 f_2 (\overline{J_{-\mu_3} f_3}), \\ J_m(f_1 \overline{f_2} f_3) &= \frac{\mu_1}{m} (J_{\mu_1} f_1) \overline{f_2} f_3 + \frac{\mu_2}{m} f_1 (\overline{J_{-\mu_2} f_2}) f_3 + \frac{\mu_3}{m} f_1 \overline{f_2} (\overline{J_{-\mu_3} f_3}), \\ J_m(\overline{f_1} f_2 f_3) &= \frac{\mu_1}{m} (\overline{J_{-\mu_1} f_1}) f_2 f_3 + \frac{\mu_2}{m} \overline{f_1} (\overline{J_{-\mu_2} f_2}) f_3 + \frac{\mu_3}{m} \overline{f_1} f_2 (\overline{J_{-\mu_3} f_3}) \end{aligned}$$

for smooth \mathbb{C} -valued functions f_1, f_2 and f_3 .

Proof. We set $\theta = x^2/(2t)$. It follows from (3.1) that

$$\begin{aligned} m J_m(f_1 f_2 \overline{f_3}) &= it e^{i(\mu_1 + \mu_2 + \mu_3)\theta} \partial_x \left\{ (e^{-i\mu_1\theta} f_1) (e^{-i\mu_2\theta} f_2) (\overline{e^{i\mu_3\theta} f_3}) \right\} \\ &= \left(it e^{i\mu_1\theta} \partial_x (e^{-i\mu_1\theta} f_1) \right) f_2 \overline{f_3} + f_1 \left(it e^{i\mu_2\theta} \partial_x (e^{-i\mu_2\theta} f_2) \right) \overline{f_3} - f_1 f_2 \left(it e^{-i\mu_3\theta} \partial_x (e^{i\mu_3\theta} f_3) \right) \\ &= (\mu_1 J_{\mu_1} f_1) f_2 \overline{f_3} + f_1 (\mu_2 J_{\mu_2} f_2) \overline{f_3} + f_1 f_2 (\mu_3 \overline{J_{-\mu_3} f_3}), \end{aligned}$$

which gives the second identity. The other three identities can be shown in the same way. \square

Remark 3.1. If we do not assume $m = \mu_1 + \mu_2 + \mu_3$, we have

$$\begin{aligned} J_m(f_1 f_2 f_3) &= \frac{\mu_1}{\mu_1 + \mu_2 + \mu_3} (J_{\mu_1} f_1) f_2 f_3 + \frac{\mu_2}{\mu_1 + \mu_2 + \mu_3} f_1 (J_{\mu_2} f_2) f_3 + \frac{\mu_3}{\mu_1 + \mu_2 + \mu_3} f_1 f_2 (J_{\mu_3} f_3) \\ &\quad + it \left(\frac{1}{m} - \frac{1}{\mu_1 + \mu_2 + \mu_3} \right) \partial_x (f_1 f_2 f_3), \end{aligned}$$

and so on. The last term implies a loss of time-decay in general. (The situation is worse if $\mu_1 + \mu_2 + \mu_3 = 0$.)

Lemma 3.2. *Let m, μ_1, μ_2 be non-zero real constants. We have*

$$\partial_x(f_1 f_2 f_3) = \frac{m}{\mu_1}(\partial_x f_1) f_2 f_3 + \frac{R_1}{t} \quad (3.2)$$

and

$$\partial_x^2(f_1 f_2 f_3) = \frac{m^2}{\mu_1 \mu_2}(\partial_x f_1)(\partial_x f_2) f_3 + \frac{R_2}{t}, \quad (3.3)$$

where $R_1 = -imJ_m(f_1 f_2 f_3) + im(J_{\mu_1} f_1) f_2 f_3$ and

$$R_2 = -\frac{im^2}{\mu_1} J_m[(\partial_x f_1) f_2 f_3] + \frac{im^2}{\mu_1} (\partial_x f_1)(J_{\mu_2} f_2) f_3 + \partial_x R_1.$$

Remark 3.2. We do not assume any relations among μ_1, μ_2 and m in Lemma 3.2.

Proof. From the relation $\frac{1}{m}\partial_x - \frac{1}{it}J_m = i\frac{x}{t}$, we see that

$$\frac{1}{m}\partial_x(f_1 f_2 f_3) - \frac{1}{it}J_m(f_1 f_2 f_3) = i\frac{x}{t}f_1 f_2 f_3 = \left(\frac{1}{\mu_1}\partial_x f_1 - \frac{1}{it}J_{\mu_1} f_1\right) f_2 f_3,$$

which yields (3.2). We also have (3.3) by using (3.2) twice. \square

Next we set

$$(\mathcal{U}_m(t)\phi)(x) := e^{i\frac{t}{2m}\partial_x^2}\phi(x) = \sqrt{\frac{|m|}{2\pi t}} e^{-i\frac{\pi}{4}\text{sgn}(m)} \int_{\mathbb{R}} e^{im\frac{(x-y)^2}{2t}} \phi(y) dy$$

for $m \in \mathbb{R} \setminus \{0\}$ and $t > 0$. We also introduce the scaled Fourier transform \mathcal{F}_m by

$$(\mathcal{F}_m\phi)(\xi) := |m|^{1/2} e^{-i\frac{\pi}{4}\text{sgn}(m)} \hat{\phi}(m\xi) = \sqrt{\frac{|m|}{2\pi}} e^{-i\frac{\pi}{4}\text{sgn}(m)} \int_{\mathbb{R}} e^{-imy\xi} \phi(y) dy,$$

as well as auxiliary operators

$$(\mathcal{M}_m(t)\phi)(x) := e^{im\frac{x^2}{2t}}\phi(x), \quad (\mathcal{D}(t)\phi)(x) := \frac{1}{\sqrt{t}}\phi\left(\frac{x}{t}\right), \quad \mathcal{W}_m(t)\phi := \mathcal{F}_m\mathcal{M}_m(t)\mathcal{F}_m^{-1}\phi,$$

so that \mathcal{U}_m can be decomposed into $\mathcal{U}_m = \mathcal{M}_m\mathcal{D}\mathcal{F}_m\mathcal{M}_m = \mathcal{M}_m\mathcal{D}\mathcal{W}_m\mathcal{F}_m$. The following lemma is well known (see e.g., [1], [14]).

Lemma 3.3. *Let m be a non-zero real constant. We have*

$$\|\phi - \mathcal{M}_m\mathcal{D}\mathcal{F}_m\mathcal{U}_m^{-1}\phi\|_{L^\infty} \leq Ct^{-3/4}(\|\phi\|_{L^2} + \|\mathcal{J}_m\phi\|_{L^2})$$

and

$$\|\phi\|_{L^\infty} \leq t^{-1/2}\|\mathcal{F}_m\mathcal{U}_m^{-1}\phi\|_{L^\infty} + Ct^{-3/4}(\|\phi\|_{L^2} + \|\mathcal{J}_m\phi\|_{L^2})$$

for $t \geq 1$.

Proof. For the convenience of the readers, we give the proof. By the relation $J_m = \mathcal{U}_m x \mathcal{U}_m^{-1}$, we see that

$$\|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{H^1} \leq C \|\mathcal{U}_m^{-1} \phi\|_{H^{0,1}} \leq C(\|\phi\|_{L^2} + \|\mathcal{J}_m \phi\|_{L^2}).$$

Also it follows from the inequalities $\|\phi\|_{L^\infty} \leq \sqrt{2} \|\phi\|_{L^2}^{1/2} \|\partial_x \phi\|_{L^2}^{1/2}$ and $|e^{i\theta} - 1| \leq C|\theta|^{1/2}$ that

$$\begin{aligned} \|(\mathcal{W}_m^{\pm 1} - 1)\phi\|_{L^\infty} &\leq C \|(\mathcal{M}_m^{\pm 1} - 1)\mathcal{F}_m^{-1} \phi\|_{L^2}^{1/2} \|\partial_x (\mathcal{W}_m^{\pm 1} - 1)\phi\|_{L^2}^{1/2} \\ &\leq C(t^{-1/2} \|\mathcal{F}_m^{-1} \phi\|_{H^{0,1}})^{1/2} \|\partial_x \phi\|_{L^2}^{1/2} \\ &\leq Ct^{-1/4} \|\phi\|_{H^1}. \end{aligned} \tag{3.4}$$

Combining with the inequalities obtained above, we have

$$\begin{aligned} \|\phi - \mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} &= \|\mathcal{M}_m \mathcal{D} (\mathcal{W}_m - 1) \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \\ &\leq t^{-1/2} \|(\mathcal{W}_m - 1) \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \\ &\leq Ct^{-3/4} \|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{H^1} \\ &\leq Ct^{-3/4} (\|\phi\|_{L^2} + \|\mathcal{J}_m \phi\|_{L^2}). \end{aligned}$$

Using the result derived above, we also have

$$\begin{aligned} \|\phi\|_{L^\infty} &\leq \|\mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} + \|\phi - \mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \\ &\leq t^{-1/2} \|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} + Ct^{-3/4} (\|\phi\|_{L^2} + \|\mathcal{J}_m \phi\|_{L^2}). \end{aligned}$$

□

Lemma 3.4. *Let m be a non-zero real constant. We have*

$$\|\mathcal{F}_m \mathcal{U}_m^{-1} (f_1 f_2 f_3)\|_{L^\infty} \leq C \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^\infty}.$$

Proof. By the relation $\mathcal{F}_m \mathcal{U}_m^{-1} = \mathcal{W}_m^{-1} \mathcal{D}^{-1} \mathcal{M}_m^{-1}$ and the estimate $\|\mathcal{W}_m^{-1} \phi\|_{L^\infty} \leq Ct^{1/2} \|\phi\|_{L^1}$, we have

$$\begin{aligned} \|\mathcal{F}_m \mathcal{U}_m^{-1} (f_1 f_2 f_3)\|_{L^\infty} &\leq Ct^{1/2} \|\mathcal{D}^{-1} \mathcal{M}_m^{-1} (f_1 f_2 f_3)\|_{L^1} \\ &\leq Ct^{1/2} \cdot t^{-1} \|(\mathcal{D}^{-1} f_1)(\mathcal{D}^{-1} f_2)(\mathcal{D}^{-1} \mathcal{M}_m^{-1} f_3)\|_{L^1} \\ &\leq Ct^{-1/2} \|\mathcal{D}^{-1} f_1\|_{L^2} \|\mathcal{D}^{-1} f_2\|_{L^2} \|\mathcal{D}^{-1} \mathcal{M}_m^{-1} f_3\|_{L^\infty} \\ &= Ct^{-1/2} \|f_1\|_{L^2} \|f_2\|_{L^2} \cdot t^{1/2} \|f_3\|_{L^\infty} \\ &= C \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^\infty}. \end{aligned}$$

□

We deduce the following proposition from Lemmas 3.1–3.4, which will play the key role in Section 5.2.

Proposition 3.1. *Suppose that the condition (a) is satisfied. For a \mathbb{C}^N -valued function $u = (u_j(t, x))_{j \in I_N}$, we set $\alpha_j(t, \xi) = \mathcal{F}_{m_j}[\mathcal{U}_{m_j}(t)^{-1}u_j(t, \cdot)](\xi)$ and $\alpha = (\alpha_j(t, \xi))_{j \in I_N}$. Then we have*

$$\left\| \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} \left[\partial_x^l F_j(u, \partial_x u) \right] - \frac{(im_j \xi)^l}{t} p_j(\xi; \alpha) \right\|_{L_\xi^\infty} \leq \frac{C}{t^{5/4}} \sum_{k=1}^N (\|u_k(t)\|_{H^3} + \|J_{m_k} u_k(t)\|_{H^2})^3$$

for $j \in I_N$, $l \in \{0, 1, 2\}$ and $t \geq 1$.

Proof. For simplicity of exposition, we treat only the case where $F_j = (\partial_x u_1)(\overline{\partial_x u_2})(\partial_x u_3)$ with $m_j = m_1 - m_2 + m_3$. The general case can be shown in the same way.

We set $\alpha_k^{(s)} = (im_k \xi)^s \alpha_k$ for $s \in \mathbb{Z}_{\geq 0}$, so that

$$\partial_x^s u_k = \mathcal{U}_{m_k} \mathcal{F}_{m_k}^{-1} \alpha_k^{(s)} = \mathcal{M}_{m_k} \mathcal{D} \mathcal{W}_{m_k} \alpha_k^{(s)}, \quad \partial_x^s \overline{u_k} = \mathcal{U}_{-m_k} \mathcal{F}_{-m_k}^{-1} \overline{\alpha_k^{(s)}}.$$

Remark that

$$p_j(\xi; \alpha) = (im_1 \xi)(-im_2 \xi)(im_3 \xi) \alpha_1 \overline{\alpha_2} \alpha_3 = \alpha_1^{(1)} \overline{\alpha_2^{(1)}} \alpha_3^{(1)}.$$

Now we consider the simplest case $l = 0$. By the factorization of \mathcal{U}_{m_j} and the condition $m_j = m_1 - m_2 + m_3$, we have

$$\begin{aligned} \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} F_j &= \mathcal{W}_{m_j}^{-1} \mathcal{D}^{-1} \mathcal{M}_{m_j}^{-1} \left[(\mathcal{M}_{m_1} \mathcal{D} \mathcal{W}_{m_1} \alpha_1^{(1)}) (\mathcal{M}_{-m_2} \mathcal{D} \mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{M}_{m_3} \mathcal{D} \mathcal{W}_{m_3} \alpha_3^{(1)}) \right] \\ &= \frac{1}{t} \mathcal{W}_{m_j}^{-1} \left[(\mathcal{W}_{m_1} \alpha_1^{(1)}) (\mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{W}_{m_3} \alpha_3^{(1)}) \right] \\ &= \frac{1}{t} p_j(\xi; \alpha) + \frac{1}{t} r_0, \end{aligned}$$

where

$$r_0 = \mathcal{W}_{m_j}^{-1} \left[(\mathcal{W}_{m_1} \alpha_1^{(1)}) (\mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{W}_{m_3} \alpha_3^{(1)}) \right] - \alpha_1^{(1)} \overline{\alpha_2^{(1)}} \alpha_3^{(1)}.$$

Since we can rewrite it as

$$\begin{aligned} r_0 &= (\mathcal{W}_{m_j}^{-1} - 1) \left[(\mathcal{W}_{m_1} \alpha_1^{(1)}) (\mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{W}_{m_3} \alpha_3^{(1)}) \right] + \{(\mathcal{W}_{m_1} - 1) \alpha_1^{(1)}\} (\mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{W}_{m_3} \alpha_3^{(1)}) \\ &\quad + \alpha_1^{(1)} \{(\mathcal{W}_{-m_2} - 1) \overline{\alpha_2^{(1)}}\} (\mathcal{W}_{m_3} \alpha_3^{(1)}) + \alpha_1^{(1)} \overline{\alpha_2^{(1)}} \{(\mathcal{W}_{m_3} - 1) \alpha_3^{(1)}\}, \end{aligned}$$

we can apply (3.4) and the Sobolev imbedding $H^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ to obtain

$$\|r_0\|_{L^\infty} \leq C t^{-1/4} \|u_1\|_{H^2} \|u_2\|_{H^2} \|u_3\|_{H^2}.$$

Next we consider the case of $l = 1$. By (3.2) with $m = m_j$, $\mu = m_1$, $f_1 = \partial_x u_1$, $f_2 = \overline{\partial_x u_2}$, $f_3 = \partial_x u_3$, we have

$$\partial_x F_j = \frac{m_j}{m_1} (\partial_x^2 u_1) (\overline{\partial_x u_2}) (\partial_x u_3) + \frac{R_1}{t}, \quad (3.5)$$

where

$$R_1 = -im_j J_{m_j} \left[(\partial_x u_1) (\overline{\partial_x u_2}) (\partial_x u_3) \right] + im_j (J_{m_1} \partial_x u_1) (\overline{\partial_x u_2}) (\partial_x u_3).$$

By applying Lemma 3.1 to the first term and using Lemma 3.4, we see that

$$\begin{aligned} \|\mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} R_1\|_{L^\infty} &\leq C \|J_{m_1} \partial_x u_1\|_{L^2} \|\partial_x u_2\|_{L^2} \|\partial_x u_3\|_{L^\infty} \\ &\quad + C \|\partial_x u_1\|_{L^2} \|J_{m_2} \partial_x u_2\|_{L^2} \|\partial_x u_3\|_{L^\infty} \\ &\quad + C \|\partial_x u_1\|_{L^2} \|\partial_x u_2\|_{L^\infty} \|J_{m_3} \partial_x u_3\|_{L^2} \\ &\leq \frac{C}{t^{1/2}} \sum_{k=1}^3 (\|u_k\|_{H^1} + \|J_{m_k} u_k\|_{H^1})^3, \end{aligned} \quad (3.6)$$

where we have used the inequality $\|\phi\|_{L^\infty} \leq Ct^{-1/2} \|\phi\|_{L^2}^{1/2} \|J_m \phi\|_{L^2}^{1/2}$ and the commutation relation $[\partial_x, J_m] = 1$ in the last line. As for the first term of (3.5), similar computations as in the previous case lead to

$$\begin{aligned} \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} \left[(\partial_x^2 u_1) (\overline{\partial_x u_2}) (\partial_x u_3) \right] &= \frac{1}{t} \mathcal{W}_{m_j}^{-1} \left[(\mathcal{W}_{m_1} \alpha_1^{(2)}) (\mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{W}_{m_3} \alpha_3^{(1)}) \right] \\ &= \frac{1}{t} \alpha_1^{(2)} \overline{\alpha_2^{(1)}} \alpha_3^{(1)} + \frac{r_1}{t}, \end{aligned}$$

where

$$r_1 = \mathcal{W}_{m_j}^{-1} \left[(\mathcal{W}_{m_1} \alpha_1^{(2)}) (\mathcal{W}_{-m_2} \overline{\alpha_2^{(1)}}) (\mathcal{W}_{m_3} \alpha_3^{(1)}) \right] - \alpha_1^{(2)} \overline{\alpha_2^{(1)}} \alpha_3^{(1)}.$$

This can be estimated as follows:

$$\|r_1\|_{L^\infty} \leq Ct^{-1/4} \|\partial_x u_1\|_{H^2} \|u_2\|_{H^2} \|u_3\|_{H^2}.$$

Moreover, we observe that

$$\frac{im_j \xi}{t} p_j(\xi; \alpha) = \frac{m_j}{m_1} \frac{im_1 \xi}{t} \alpha_1^{(1)} \overline{\alpha_2^{(1)}} \alpha_3^{(1)} = \frac{m_j}{m_1} \cdot \frac{1}{t} \alpha_1^{(2)} \overline{\alpha_2^{(1)}} \alpha_3^{(1)}.$$

Piecing them together, we arrive at

$$\begin{aligned} \left\| \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} \partial_x F_j - \frac{im_j \xi}{t} p_j(\xi; \alpha) \right\|_{L_\xi^\infty} &= \frac{1}{t} \left\| \frac{m_j}{m_1} r_1 + \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} R_1 \right\|_{L^\infty} \\ &\leq \frac{C}{t} (\|r_1\|_{L^\infty} + \|\mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} R_1\|_{L^\infty}) \\ &\leq \frac{C}{t^{5/4}} \sum_{k=1}^3 (\|u_k(t, \cdot)\|_{H^3} + \|J_{m_k} u_k(t, \cdot)\|_{H^1})^3, \end{aligned}$$

as desired. Finally we consider the case of $l = 2$. By (3.3) with $m = m_j$, $\mu_1 = m_1$ and $\mu_2 = -m_2$, we have

$$\partial_x^2 F_j = \frac{m_j^2}{-m_1 m_2} (\partial_x^2 u_1) (\overline{\partial_x^2 u_2}) (\partial_x u_3) + \frac{R_2}{t},$$

where

$$R_2 = -\frac{im_j^2}{m_1}J_{m_j}\left[(\partial_x^2 u_1)(\overline{\partial_x u_2})(\partial_x u_3)\right] + \frac{im_j^2}{m_1}(\partial_x^2 u_1)(\overline{J_{m_2}\partial_x u_2})(\partial_x u_3) \\ - im_j\partial_x J_{m_j}\left[(\partial_x u_1)(\overline{\partial_x u_2})(\partial_x u_3)\right] + im_j\partial_x \left[(J_{m_1}\partial_x u_1)(\overline{\partial_x u_2})(\partial_x u_3)\right].$$

As in the derivation of (3.6), we see that

$$\|\mathcal{F}_{m_j}\mathcal{U}_{m_j}^{-1}R_2\|_{L^\infty} \leq \frac{C}{t^{1/2}} \sum_{k=1}^3 (\|u_k\|_{H^2} + \|J_{m_k}u_k\|_{H^2})^3.$$

Similarly to the previous cases, we can also show that

$$\mathcal{F}_{m_j}\mathcal{U}_{m_j}^{-1}\left[(\partial_x^2 u_1)(\overline{\partial_x^2 u_2})(\partial_x u_3)\right] = \frac{1}{t}\mathcal{W}_{m_j}^{-1}\left[(\mathcal{W}_{m_1}\alpha_1^{(2)})(\mathcal{W}_{-m_2}\overline{\alpha_2^{(2)}})(\mathcal{W}_{m_3}\alpha_3^{(1)})\right] \\ = \frac{-m_1m_2}{m_j^2}\frac{(im_j\xi)^2}{t}p_j(\xi; \alpha) + \frac{r_2}{t},$$

where

$$r_2 = \mathcal{W}_{m_j}^{-1}\left[(\mathcal{W}_{m_1}\alpha_1^{(2)})(\mathcal{W}_{-m_2}\overline{\alpha_2^{(2)}})(\mathcal{W}_{m_3}\alpha_3^{(1)})\right] - \alpha_1^{(2)}\overline{\alpha_2^{(2)}}\alpha_3^{(1)}.$$

Note that

$$\|r_2\|_{L^\infty} \leq Ct^{-1/4}\|\partial_x u_1\|_{H^2}\|\partial_x u_2\|_{H^2}\|u_3\|_{H^2}.$$

Therefore we have

$$\left\|\mathcal{F}_{m_j}\mathcal{U}_{m_j}^{-1}\partial_x^2 F_j - \frac{(im_j\xi)^2}{t}p_j(\xi, \alpha)\right\|_{L_\xi^\infty} \leq \frac{C}{t}(\|r_2\|_{L^\infty} + \|\mathcal{F}_{m_j}\mathcal{U}_{m_j}^{-1}R_2\|_{L^\infty}) \\ \leq \frac{C}{t^{5/4}} \sum_{k=1}^3 (\|u_k(t, \cdot)\|_{H^3} + \|J_{m_k}u_k(t, \cdot)\|_{H^2})^3,$$

which completes the proof. \square

4 Smoothing effect

In this section, we recall smoothing properties of the linear Schrödinger equations. As is well known, the standard energy method causes a derivative loss when the nonlinear term involves derivatives of the unknown functions. Smoothing effect is a useful tool to overcome this obstacle. Among various kinds of such techniques, we will follow the approach of [2]. Let \mathcal{H} be the Hilbert transform, that is,

$$\mathcal{H}\psi(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\psi(y)}{x-y} dy.$$

With a non-negative weight function $\Phi(x)$ and a non-zero real constant m , let us also define the operator $S_{\Phi,m}$ by

$$S_{\Phi,m}\psi(x) := \left\{ \cosh\left(\int_{-\infty}^x \Phi(y)dy\right) \right\} \psi(x) - i \operatorname{sgn}(m) \left\{ \sinh\left(\int_{-\infty}^x \Phi(y)dy\right) \right\} \mathcal{H}\psi(x).$$

Note that $S_{\Phi,m}$ is L^2 -automorphism and that both $\|S_{\Phi,m}\|_{L^2 \rightarrow L^2}$, $\|S_{\Phi,m}^{-1}\|_{L^2 \rightarrow L^2}$ are dominated by $C \exp(\|\Phi\|_{L^1})$. This operator enables us to gain the half-derivative $|\partial_x|^{1/2}$. More precisely, we have the following:

Lemma 4.1. *Let m, μ_1, \dots, μ_N be non-zero real constants. Let v be a \mathbb{C} -valued smooth function of (t, x) , and let $w = (w_j)_{j \in I_N}$ be a \mathbb{C}^N -valued smooth function of (t, x) . We set $\Phi = \eta(|w|^2 + |\partial_x w|^2)$ with $\eta \geq 1$, and $S = S_{\Phi(t, \cdot), m}$. Then we have*

$$\begin{aligned} \frac{d}{dt} \|Sv(t)\|_{L^2}^2 + \frac{1}{|m|} \int_{\mathbb{R}} \Phi(t, x) \left| S|\partial_x|^{1/2} v(t, x) \right|^2 dx \\ \leq 2 \left| \langle Sv(t), S\mathcal{L}_m v(t) \rangle_{L^2} \right| + CB(t) \|v(t)\|_{L^2}^2, \end{aligned}$$

where

$$B(t) = e^{C\eta\|w\|_{H^1}^2} \left\{ \eta \|w(t)\|_{W^{2,\infty}}^2 + \eta^3 \|w(t)\|_{W^{1,\infty}}^6 + \eta \sum_{k \in I_N} \|w_k(t)\|_{H^1} \|\mathcal{L}_{\mu_k} w_k(t)\|_{H^1} \right\}$$

and the constant C is independent of η . We denote by $W^{s,\infty}$ the L^∞ -based Sobolev space of order $s \in \mathbb{Z}_{\geq 0}$.

This lemma is essentially the same as Lemma 2.1 in [2], although we need slight modifications to fit for our purpose. For the convenience of the readers, we will give the proof of this lemma in the appendix.

By using Lemma 4.1 combined with the following auxiliary lemma, we can get rid of the derivative loss coming from the nonlinear terms.

Lemma 4.2. *Let m_1, \dots, m_N be non-zero real constants. Let $v = (v_j)_{j \in I_N}$, $w = (w_j)_{j \in I_N}$ be \mathbb{C}^N -valued smooth functions of $x \in \mathbb{R}$. Suppose that $q_{1,jk}$ and $q_{2,jk}$ are quadratic homogeneous polynomials in $(w, \partial_x w, \overline{w}, \overline{\partial_x w})$. We set $\Phi = \eta(|w|^2 + |\partial_x w|^2)$ with $\eta \geq 1$, and $S = S_{\Phi(t, \cdot), m}$ with $\eta \geq 1$, and $S_j = S_{\Phi, m_j}$ for $j \in I_N$. Then we have*

$$\begin{aligned} \sum_{j,k \in I_N} \left(\left| \langle S_j v_j, S_j(q_{1,jk} \partial_x v_k) \rangle_{L^2} \right| + \left| \langle S_j v_j, S_j(q_{2,jk} \overline{\partial_x v_k}) \rangle_{L^2} \right| \right) \\ \leq \frac{C}{\eta} e^{C\eta\|w\|_{H^1}^2} \sum_{k \in I_N} \int_{\mathbb{R}} \Phi(x) \left| S_k |\partial_x|^{1/2} v_k(x) \right|^2 dx \\ + C e^{C\eta\|w\|_{H^1}^2} (1 + \eta^2 \|w\|_{H^1}^4 + \eta^2 \|w\|_{W^{1,\infty}}^4) \|w\|_{W^{2,\infty}}^2 \|v\|_{L^2}^2, \end{aligned}$$

where the constant C is independent of η .

We skip the proof of Lemma 4.2 because this is nothing more than a paraphrase of Lemma 2.3 in [2].

5 A priori estimate

Let $T \in (0, +\infty]$, and let $u = (u_j)_{1 \leq j \leq N} \in C([0, T]; H^3 \cap H^{2,1})$ be a solution to (1.1) for $t \in [0, T]$. As in Section 3, we set $\alpha_j(t, \xi) = \mathcal{F}_{m_j} \left[\mathcal{U}_{m_j}^{-1} u_j(t, \cdot) \right](\xi)$, $\alpha(t, \xi) = (\alpha_j(t, \xi))_{j \in I_N}$, and define

$$E(T) = \sup_{0 \leq t < T} \sum_{j \in I_N} \left[(1+t)^{-\frac{\gamma}{3}} \left(\|u_j(t)\|_{H^3} + \|J_{m_j} u_j(t)\|_{H^2} \right) + \sup_{\xi \in \mathbb{R}} \left(\langle \xi \rangle^2 |\alpha_j(t, \xi)| \right) \right]$$

with $\gamma > 0$. The goal of this section is to show the following:

Lemma 5.1. *Assume the conditions (a) and (b₀) are satisfied. Let $\gamma \in (0, 1/4)$. There exist positive constants ε_1 and K such that*

$$E(T) \leq \varepsilon^{2/3} \tag{5.1}$$

implies

$$E(T) \leq K\varepsilon,$$

provided that $\varepsilon = \|\varphi\|_{H^3 \cap H^{2,1}} \leq \varepsilon_1$.

The proof of this lemma will be divided into two parts.

5.1 L^2 -estimates

In the first part, we consider the bounds for $\|u_j(t)\|_{H^3}$ and $\|J_{m_j} u_j(t)\|_{H^2}$. It is enough to show

$$\sum_{j \in I_N} \sum_{l=0}^1 \|J_{m_j}^l u_j(t)\|_{L^2} \leq C\varepsilon + C\varepsilon^2(1+t)^{\gamma/3} \tag{5.2}$$

and

$$\sum_{j \in I_N} \sum_{l=0}^1 \|\partial_x^{3-l} J_{m_j}^l u_j(t)\|_{L^2}^2 \leq C\varepsilon^2(1+t)^{2\gamma/3} \tag{5.3}$$

for $t \in [0, T]$ under the assumption (5.1). First we remark that (5.1) implies a rough H^1 -bound

$$\|u_j(t)\|_{H^1} \leq C\|\alpha_j(t)\|_{H^{0,1}} \leq C \left(\int_{\mathbb{R}} \frac{d\xi}{\langle \xi \rangle^2} \right)^{1/2} \sup_{\xi \in \mathbb{R}} (\langle \xi \rangle^2 |\alpha_j(t, \xi)|) \leq C\varepsilon^{2/3} \tag{5.4}$$

for $t \in [0, T)$. We also deduce from (5.1) that

$$\|u_j(t)\|_{W^{2,\infty}} \leq \frac{C\varepsilon^{2/3}}{(1+t)^{1/2}}$$

for $t \in [0, T)$. Indeed, it follows from Lemma 3.3 and the relation $[\partial_x, J_{m_j}] = 1$ that

$$\|u_j(t)\|_{W^{2,\infty}} \leq \frac{C}{t^{1/2}} \sup_{\xi \in \mathbb{R}} |\langle \xi \rangle^2 \alpha_j(t, \xi)| + \frac{C}{t^{3/4}} (\|u_j(t)\|_{H^2} + \|J_{m_j} u_j(t)\|_{H^2}) \leq \frac{C\varepsilon^{2/3}}{t^{1/2}}$$

for $t \geq 1$, and $H^1(\mathbb{R}^1) \hookrightarrow L^\infty(\mathbb{R}^1)$ yields $\|u_j(t)\|_{W^{2,\infty}} \leq C\|u_j(t)\|_{H^3} \leq C\varepsilon^{2/3}$ for $t \leq 1$.

Now we consider the easier estimate (5.2). It follows from the standard energy method that

$$\begin{aligned} \frac{d}{dt} \|u_j(t)\|_{L^2} &\leq \|F_j(u(t), \partial_x u(t))\|_{L^2} \\ &\leq C \|u(t)\|_{W^{1,\infty}}^2 \|u(t)\|_{H^1} \\ &\leq C \left(\frac{\varepsilon^{2/3}}{(1+t)^{1/2}} \right)^2 \cdot C\varepsilon^{2/3} \\ &\leq \frac{C\varepsilon^2}{1+t}. \end{aligned}$$

Also we see from Lemma 3.1 that

$$\mathcal{L}_{m_j} J_{m_j} u_j = \sum_{k \in I_N} \left(q_{1,jk} J_{m_k} \partial_x u_k + q_{2,jk} \overline{J_{m_k} \partial_x u_k} + q_{3,jk} J_{m_k} u_k + q_{4,jk} \overline{J_{m_k} u_k} \right),$$

where $q_{1,jk}, \dots, q_{4,jk}$ are quadratic homogeneous polynomials in $(u, \partial_x u, \overline{u}, \overline{\partial_x u})$. Then the standard energy method again implies

$$\frac{d}{dt} \|J_{m_j} u_j(t)\|_{L^2} \leq C \|u\|_{W^{1,\infty}}^2 \sum_{k \in I_N} (\|u_k\|_{H^1} + \|J_{m_k} u_k\|_{H^1}) \leq \frac{C\varepsilon^2}{(1+t)^{1-\gamma/3}}.$$

These lead to (5.2).

Next we consider (5.3). We set $v_{jl} = \partial_x^{3-l} J_{m_j}^l u_j$ for $l \in \{0, 1\}$ and $j \in I_N$. We apply Lemma 4.1 with $m = m_j$, $\mu_k = m_k$, $v = v_{jl}$, $w = u$, $\eta = \varepsilon^{-2/3}$. Then we obtain

$$\begin{aligned} &\frac{d}{dt} \|S_j v_{jl}(t)\|_{L^2}^2 + \frac{1}{|m_j|} \int_{\mathbb{R}} \Phi(t, x) \left| S_j |\partial_x|^{1/2} v_{jl}(t) \right|^2 dx \\ &\leq 2 \left| \langle S_j v_{jl}, S_j \partial_x^{3-l} J_{m_j}^l F_j(u, \partial_x u) \rangle_{L^2} \right| + CB(t) \|v_{jl}(t)\|_{L^2}^2, \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} B(t) &= e^{\frac{C}{\varepsilon^{2/3}} \|u\|_{H^1}^2} \left(\varepsilon^{-2/3} \|u\|_{W^{2,\infty}}^2 + \varepsilon^{-2} \|u\|_{W^{1,\infty}}^6 + \varepsilon^{-2/3} \sum_{k \in I_N} \|u_k\|_{H^1} \|F_k(u, \partial_x u)\|_{H^1} \right) \\ &\leq \frac{C\varepsilon^{2/3}}{1+t}. \end{aligned}$$

To estimate the first term of the right-hand side of (5.5), we use Lemma 3.1 and the usual Leibniz rule to split $\partial_x^{3-l} J_{m_j}^l F_j(u, \partial_x u)$ into the following form:

$$\sum_{k \in I_N} \left(g_{1,jkl} \partial_x v_{kl} + g_{2,jkl} \overline{\partial_x v_{kl}} \right) + h_{jl},$$

where $g_{1,jkl}$ and $g_{2,jkl}$ are quadratic homogeneous polynomials in $(u, \partial_x u, \bar{u}, \overline{\partial_x u})$, and h_{jl} is a cubic term satisfying

$$\|h_{jl}\|_{L^2} \leq C \|u(t)\|_{W^{2,\infty}}^2 \sum_{k \in I_N} (\|u_k(t)\|_{H^3} + \|J_{m_k} u_k(t)\|_{H^2}) \leq \frac{C\varepsilon^2}{(1+t)^{1-\gamma/3}}.$$

Then Lemma 4.2 and the L^2 -automorphism of S_j lead to

$$\begin{aligned} & \sum_{j \in I_N} \left| \langle S_j v_{jl}, S_j \partial_x^{3-l} J_{m_j}^l F_j(u, \partial_x u) \rangle_{L^2} \right| \\ & \leq \sum_{j,k \in I_N} \left(|\langle S_j v_{jl}, S_j (g_{1,jkl} \partial_x v_{kl}) \rangle_{L^2}| + |\langle S_j v_{jl}, S_j (g_{2,jkl} \overline{\partial_x v_{kl}}) \rangle_{L^2}| \right) + \sum_{j \in I_N} \|S_j v_{jl}\|_{L^2} \|S_j h_{jl}\|_{L^2} \\ & \leq C\varepsilon^{2/3} e^{\frac{C}{\varepsilon^{2/3}} \|u\|_{H^1}^2} \sum_{k \in I_N} \int_{\mathbb{R}} \Phi(t, x) \left| S_k |\partial_x|^{1/2} v_{kl}(t, x) \right|^2 dx \\ & \quad + C e^{\frac{C}{\varepsilon^{2/3}} \|u\|_{H^1}^2} (1 + \varepsilon^{-4/3} \|u\|_{H^1}^4 + \varepsilon^{-4/3} \|u\|_{W^{1,\infty}}^4) \|u\|_{W^{2,\infty}}^2 \sum_{k \in I_N} \|v_{kl}\|_{L^2}^2 \\ & \quad + C e^{\frac{C}{\varepsilon^{2/3}} \|u\|_{H^1}^2} \sum_{j \in I_N} \|v_{jl}\|_{L^2} \|h_{jl}\|_{L^2} \\ & \leq C_0 \varepsilon^{2/3} \sum_{k \in I_N} \int_{\mathbb{R}} \Phi(t, x) \left| S_k |\partial_x|^{1/2} v_{kl}(t, x) \right|^2 dx + \frac{C\varepsilon^{8/3}}{(1+t)^{1-2\gamma/3}} \end{aligned}$$

with some positive constant C_0 not depending on ε . Summing up, we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{j \in I_N} \|S_j v_{jl}(t)\|_{L^2}^2 & \leq \sum_{k \in I_N} \left(2C_0 \varepsilon^{2/3} - \frac{1}{|m_k|} \right) \int_{\mathbb{R}} \Phi(t, x) \left| S_k |\partial_x|^{1/2} v_{kl}(t, x) \right|^2 dx \\ & \quad + \frac{C\varepsilon^{8/3}}{(1+t)^{1-2\gamma/3}} + \frac{C\varepsilon^{2/3}}{1+t} \cdot (C\varepsilon^{2/3} (1+t)^{\gamma/3})^2 \\ & \leq \frac{C\varepsilon^2}{(1+t)^{1-2\gamma/3}}, \end{aligned}$$

provided that

$$2C_0 \varepsilon^{2/3} \leq \frac{1}{\min_{1 \leq k \leq N} |m_k|}.$$

Integrating with respect to t , we have

$$\sum_{j \in I_N} \|S_j v_{jl}(t)\|_{L^2}^2 \leq C\varepsilon^2 + C\varepsilon^2 (1+t)^{2\gamma/3} \leq C\varepsilon^2 (1+t)^{2\gamma/3},$$

whence

$$\sum_{j \in I_N} \sum_{l=0}^1 \|\partial_x^{3-l} J_{m_j}^l u_j(t)\|_{L^2}^2 \leq e^{C\varepsilon^{-2/3}\|u(t)\|_{H^1}^2} \sum_{j \in I_N} \sum_{l=0}^1 \|S_j v_{jl}(t)\|_{L^2}^2 \leq C\varepsilon^2(1+t)^{2\gamma/3},$$

as required. \square

5.2 Estimates for α_j

In the second part, we are going to show $\langle \xi \rangle^2 |\alpha(t, \xi)| \leq C\varepsilon$ for $(t, \xi) \in [0, T) \times \mathbb{R}$ under the assumption (5.1). If $t \in [0, 1]$, the Sobolev imbedding yields this estimate immediately. Hence we have only to consider the case of $t \in [1, T)$. We set

$$\rho_j(t, \xi) = \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} [F_j(u, \partial_x u)] - \frac{1}{t} p_j(\xi; \alpha(t, \xi))$$

and $\rho = (\rho_j)_{j \in I_N}$, so that

$$\begin{aligned} i\partial_t \alpha_j(t, \xi) &= \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} [\mathcal{L}_{m_j} u_j] = \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} [F_j(u, \partial_x u)] \\ &= \frac{1}{t} p_j(\xi; \alpha(t, \xi)) + \rho_j(t, \xi). \end{aligned} \quad (5.6)$$

By Proposition 3.1, we have

$$\begin{aligned} |\rho_j(t, \xi)| &\leq \frac{C}{\langle \xi \rangle^2} \sum_{l=0}^2 |(im_j \xi)^l \rho_j(t, \xi)| \\ &= \frac{C}{\langle \xi \rangle^2} \sum_{l=0}^2 \left| \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} [\partial_x^l F_j(u, \partial_x u)] - \frac{(im_j \xi)^l}{t} p_j(\xi; \alpha(t, \xi)) \right| \\ &\leq \frac{C}{\langle \xi \rangle^2} \cdot \frac{C}{t^{5/4}} \left(E(T) t^{\frac{\gamma}{3}} \right)^3 \\ &\leq \frac{C\varepsilon^2}{\langle \xi \rangle^2 t^{5/4-\gamma}} \end{aligned}$$

for $t \geq 1$ and $\xi \in \mathbb{R}$, which shows that $\rho_j(t, \xi)$ has enough decay rates both in t and ξ . Now we put $\nu(t, \xi) = \sqrt{\langle \alpha(t, \xi), A\alpha(t, \xi) \rangle_{\mathbb{C}^N}}$, where A is the positive Hermitian matrix appearing in the condition (b₀). Remark that

$$\sqrt{\kappa_*} |\alpha(t, \xi)| \leq \nu(t, \xi) \leq \sqrt{\kappa^*} |\alpha(t, \xi)|,$$

where κ_* and κ^* are the smallest and largest eigenvalues of A , respectively. It follows from (b₀) that

$$\begin{aligned} \partial_t \nu(t, \xi)^2 &= 2 \operatorname{Im} \langle i\partial_t \alpha(t, \xi), A\alpha(t, \xi) \rangle_{\mathbb{C}^N} \\ &= \frac{2}{t} \operatorname{Im} \langle p(\xi; \alpha(t, \xi)), A\alpha(t, \xi) \rangle_{\mathbb{C}^N} + 2 \operatorname{Im} \langle \rho(t, \xi), A\alpha(t, \xi) \rangle_{\mathbb{C}^N} \\ &\leq 0 + C |\rho(t, \xi)| \nu(t, \xi), \end{aligned}$$

which leads to

$$\nu(t, \xi) \leq \nu(1, \xi) + C \int_1^t |\rho(\tau, \xi)| d\tau \leq \frac{C\varepsilon}{\langle \xi \rangle^2} + \frac{C\varepsilon^2}{\langle \xi \rangle^2} \int_1^\infty \frac{d\tau}{\tau^{5/4-\gamma}} \leq \frac{C\varepsilon}{\langle \xi \rangle^2},$$

Therefore we have

$$\langle \xi \rangle^2 |\alpha_j(t, \xi)| \leq C \langle \xi \rangle^2 \nu(t, \xi) \leq C\varepsilon,$$

as required. \square

6 Proof of the main theorems

Now we are in a position to prove Theorems 2.1 – 2.4.

6.1 Proof of Theorem 2.1

First let us recall the local existence theorem. For fixed $t_0 \geq 0$, let us consider the initial value problem

$$\begin{cases} \mathcal{L}_{m_j} u_j = F_j(u, \partial_x u), & t > t_0, x \in \mathbb{R}, j \in I_N, \\ u_j(t_0, x) = \psi_j(x), & x \in \mathbb{R}, j \in I_N. \end{cases} \quad (6.1)$$

Lemma 6.1. *Let $\psi = (\psi_j)_{j \in I_N} \in H^3 \cap H^{2,1}$. There exists a positive constant ε_0 , which is independent of t_0 , such that the following holds: for any $\underline{\varepsilon} \in (0, \varepsilon_0)$ and $M \in (0, \infty)$, one can choose a positive constant $\tau^* = \tau^*(\underline{\varepsilon}, M)$, which is independent of t_0 , such that (6.1) admits a unique solution $u = (u_j)_{j \in I_N} \in C([t_0, t_0 + \tau^*]; H^3 \cap H^{2,1})$, provided that*

$$\|\psi\|_{H^1} \leq \underline{\varepsilon} \quad \text{and} \quad \sum_{l=0}^1 \sum_{j \in I_N} \left\| \left(x + i \frac{t_0}{m_j} \partial_x \right)^l \psi_j \right\|_{H^{3-l}} \leq M.$$

We omit the proof of this lemma because it is standard (see e.g., Appendix of [4] for the proof of similar lemma in the quadratic nonlinear case).

Now we are going to prove the global existence by the so-called bootstrap argument. Let T^* be the supremum of all $T \in (0, \infty]$ such that the problem (1.1) admits a unique solution $u \in C([0, T]; H^3 \cap H^{2,1})$. By Lemma 6.1 with $t_0 = 0$, we have $T^* > 0$ if $\|\varphi\|_{H^1} \leq \varepsilon < \varepsilon_0$. We also set

$$T_* = \sup \{ \tau \in [0, T^*) \mid E(\tau) \leq \varepsilon^{2/3} \}.$$

Note that $T_* > 0$ because of the continuity of $[0, T^*) \ni \tau \mapsto E(\tau)$ and $\|\varphi\|_{H^3 \cap H^{2,1}} = \varepsilon \leq \frac{1}{2} \varepsilon^{2/3}$ if $\varepsilon \leq 1/8$.

We claim that $T_* = T^*$ if ε is small enough. Indeed, if $T_* < T^*$, Lemma 5.1 with $T = T_*$ yields

$$E(T_*) \leq K\varepsilon \leq \frac{1}{2} \varepsilon^{2/3}$$

for $\varepsilon \leq \varepsilon_2 := \min\{\varepsilon_1, 1/(2K)^3\}$, where K and ε_1 are mentioned in Lemma 5.1. By the continuity of $[0, T^*) \ni \tau \mapsto E(\tau)$, we can take $T^b \in (T_*, T^*)$ such that $E(T^b) \leq \varepsilon^{2/3}$, which contradicts the definition of T_* . Therefore we must have $T_* = T^*$. By using Lemma 5.1 with $T = T^*$ again, we see that

$$\sum_{l=0}^1 \sum_{j \in I_N} \|J_{m_j}^l u_j(t, \cdot)\|_{H^{3-l}} \leq K\varepsilon(1+t)^{\frac{\gamma}{3}}, \quad \sum_{j \in I_N} \sup_{\xi \in \mathbb{R}} \left(\langle \xi \rangle^2 |\alpha_j(t, \xi)| \right) \leq K\varepsilon$$

for $t \in [0, T^*)$. In particular we have

$$\sup_{t \in [0, T^*)} \|u(t)\|_{H^1} \leq C \sup_{(t, \xi) \in [0, T^*) \times \mathbb{R}} \left(\langle \xi \rangle^2 |\alpha(t, \xi)| \right) \leq C^b \varepsilon$$

with some $C^b > 0$.

Next we assume $T^* < \infty$. Then, by setting $\varepsilon_3 = \min\{\varepsilon_2, \varepsilon_0/2C^b\}$ and $M = K\varepsilon_3(1+T^*)^{\gamma/3}$, we have

$$\sup_{t \in [0, T^*)} \sum_{l=0}^1 \sum_{j \in I_N} \|J_{m_j}^l u_j(t, \cdot)\|_{H^{3-l}} \leq M$$

as well as

$$\sup_{t \in [0, T^*)} \|u(t)\|_{H^1} \leq \varepsilon_0/2 < \varepsilon_0$$

for $\varepsilon \leq \varepsilon_3$. By Lemma 6.1, there exists $\tau^* > 0$ such that (1.1) admits the solution $u \in C([0, T^* + \tau^*); H^3 \cap H^{2,1})$. This contradicts the definition of T^* , which means $T^* = +\infty$ for $\varepsilon \in (0, \varepsilon_3]$. Moreover, we have

$$\|u(t)\|_{L^2} \leq C \sup_{\xi \in \mathbb{R}} |\langle \xi \rangle \alpha(t, \xi)| \leq C\varepsilon.$$

By using Lemma 3.3 and the inequality obtained above, we also have

$$|u_j(t, x)| \leq \frac{C}{t^{1/2}} |\alpha_j(t, \xi)| + \frac{C}{t^{3/4}} (\|u_j(t)\|_{L^2} + \|J_{m_j} u_j(t)\|_{L^2}) \leq \frac{C\varepsilon}{t^{1/2}}$$

for $t \geq 1$ and $j \in I_N$. This completes the proof of Theorem 2.1. \square

6.2 Proof of Theorems 2.2 and 2.3

The proof of Theorems 2.2 and 2.3 heavily relies on the following lemma due to [6]. Note that special cases of this lemma have been used previously in [5] and [10] less explicitly.

Lemma 6.2 ([6]). *Let $C_0 > 0$, $C_1 \geq 0$, $p > 1$ and $q > 1$. Suppose that $\Psi(t)$ satisfies*

$$\frac{d\Psi}{dt}(t) \leq \frac{-C_0}{t} |\Psi(t)|^p + \frac{C_1}{t^q}$$

for $t \geq 2$. Then we have

$$\Psi(t) \leq \frac{C_2}{(\log t)^{p^*-1}}$$

for $t \geq 2$, where p^* is the Hölder conjugate of p (i.e., $1/p + 1/p^* = 1$), and

$$C_2 = \left(\frac{p^*}{C_0 p}\right)^{p^*-1} + (\log 2)^{p^*-1} \Psi(2) + \frac{C_1}{\log 2} \int_2^\infty \frac{(\log \tau)^{p^*}}{\tau^q} d\tau.$$

With $\xi \in \mathbb{R}$ fixed, we set $\Psi(t) = \langle \alpha(t, \xi), A\alpha(t, \xi) \rangle_{\mathbb{C}^N}$, where A is the positive Hermitian matrix appearing in the condition (b₁). Then we deduce from (5.6) that Ψ satisfies

$$\frac{d\Psi}{dt}(t) \leq \frac{-2C_*}{t} |\alpha(t)|^4 + C |\rho(t, \xi)| |\alpha(t, \xi)| \leq \frac{-2C_*/\kappa_*^2}{t} |\Psi(t)|^2 + \frac{C\varepsilon^3}{\langle \xi \rangle^4 t^{5/4-\gamma}}$$

for $t \geq 2$, where C_* is the positive constant appearing in the condition (b₁) and κ_* is the smallest eigenvalue of A . We also have $\Psi(2) \leq C |\alpha(2, \xi)|^2 \leq C\varepsilon^2 \langle \xi \rangle^{-4}$. So we can apply Lemma 6.2 with $p = 2$, $q = 5/4 - \gamma$ to obtain

$$|\alpha(t, \xi)|^2 \leq C\Psi(t) \leq \frac{1}{(\log t)^{2-1}} \left(\frac{\kappa_*^2}{2C_*} + \frac{C\varepsilon^2}{\langle \xi \rangle^4} \right) \leq \frac{C}{\log t}.$$

From Lemma 3.3 it follows that

$$\begin{aligned} |u_j(t, x)| &\leq \frac{C}{t^{1/2}} \sup_{\xi \in \mathbb{R}} |\alpha_j(t, \xi)| + \frac{C}{t^{3/4}} (\|u_j(t)\|_{L^2} + \|J_{m_j} u_j(t)\|_{L^2}) \\ &\leq \frac{C}{(t \log t)^{1/2}} + \frac{C\varepsilon}{t^{3/4-\gamma/3}} \\ &\leq \frac{C}{(t \log t)^{1/2}}, \end{aligned}$$

for $t \geq 2$, $x \in \mathbb{R}$ and $j \in I_N$. On the other hand, we already know that $|u(t, x)| \leq C\varepsilon(1+t)^{-1/2}$ for $t \geq 0$. Hence we arrive at

$$(1+t)(1+\varepsilon^2 \log(t+2)) |u(t, x)|^2 \leq C\varepsilon^2$$

for $t \geq 0$, which implies the desired pointwise decay estimate. By the Fatou lemma we also have

$$\limsup_{t \rightarrow +\infty} \|\alpha_j(t)\|_{L^2}^2 \leq \int_{\mathbb{R}} \limsup_{t \rightarrow +\infty} |\alpha_j(t, \xi)|^2 d\xi = 0,$$

which leads to decay of $\|u_j(t)\|_{L^2}$ as $t \rightarrow +\infty$, as stated in Theorem 2.2.

Under the stronger condition (b₂), we have

$$\frac{d\Psi}{dt}(t) \leq \frac{-2C_{**} \langle \xi \rangle^2 / \kappa_*^2}{t} |\Psi(t)|^2 + \frac{C\varepsilon^3}{\langle \xi \rangle^4 t^{5/4-\gamma}}$$

for $t \geq 2$. Therefore Lemma 6.2 again yields

$$|\alpha(t, \xi)|^2 \leq \frac{1}{\log t} \left(\frac{\kappa_*^2}{2C_{**}\langle \xi \rangle^2} + \frac{C\varepsilon^2}{\langle \xi \rangle^4} \right) \leq \frac{C}{\langle \xi \rangle^2 \log t},$$

whence

$$\|u(t)\|_{L^2} = \|\alpha(t)\|_{L^2} \leq C \sup_{\xi \in \mathbb{R}} (\langle \xi \rangle |\alpha(t, \xi)|) \leq \frac{C}{\sqrt{\log t}}$$

for $t \geq 2$. This yields Theorem 2.3. \square

6.3 Proof of Theorem 2.4

For given $\delta > 0$, we set $\gamma = \min\{\delta, 1/5\} \in (0, 1/4)$. Remember that we have already shown that

$$|\alpha_j(t, \xi)| \leq \frac{C\varepsilon}{\langle \xi \rangle^2}, \quad |\rho_j(t, \xi)| \leq \frac{C\varepsilon^2}{\langle \xi \rangle^2 t^{5/4-\gamma}}$$

for $t \geq 1$, $\xi \in \mathbb{R}$ and $j \in I_N$. These estimates allow us to define $\alpha^+ = (\alpha_j^+)_{j \in I_N} \in L^2 \cap L^\infty$ by

$$\alpha_j^+(\xi) := \alpha_j(1, \xi) - i \int_1^\infty \rho_j(t', \xi) dt'.$$

On the other hand, the condition (b₃) and (5.6) lead to

$$\alpha_j(t, \xi) = \alpha_j(1, \xi) - i \int_1^t \rho_j(t', \xi) dt',$$

whence

$$\|\alpha_j(t) - \alpha_j^+\|_{L^2 \cap L^\infty} \leq \int_t^\infty \|\rho_j(t', \cdot)\|_{L^2 \cap L^\infty} dt' \leq C\varepsilon^2 t^{-1/4+\gamma}.$$

Now we set $\varphi_j^+ := \mathcal{F}_{m_j}^{-1} \alpha_j^+$. Then we have

$$\begin{aligned} \|u_j(t) - \mathcal{U}_{m_j} \varphi_j^+\|_{L^2} &= \|\mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} u_j(t) - \mathcal{F}_{m_j} \varphi_j^+\|_{L^2} \\ &= \|\alpha_j(t) - \alpha_j^+\|_{L^2} \\ &\leq C\varepsilon^2 t^{-1/4+\gamma}. \end{aligned}$$

By Lemma 3.3 and the inequality obtained above, we also have

$$\begin{aligned} &\|u_j(t) - \mathcal{M}_{m_j} \mathcal{D} \mathcal{F}_{m_j} \varphi_j^+\|_{L^\infty} \\ &\leq \|u_j(t) - \mathcal{M}_{m_j} \mathcal{D} \mathcal{F}_{m_j} \mathcal{U}_{m_j}^{-1} u_j(t)\|_{L^\infty} + \|\mathcal{M}_{m_j} \mathcal{D}(\alpha_j(t) - \alpha_j^+)\|_{L^\infty} \\ &\leq Ct^{-3/4} (\|u_j(t)\|_{L^2} + \|J_{m_j} u_j(t)\|_{L^2}) + Ct^{-1/2} \|\alpha_j(t) - \alpha_j^+\|_{L^\infty} \\ &\leq C\varepsilon t^{-3/4+\gamma/3} + C\varepsilon^2 t^{-1/2-1/4+\gamma} \\ &\leq C\varepsilon t^{-3/4+\delta} \end{aligned}$$

for $t \geq 1$. \square

Remark 6.1. We put $\varphi_j = \varepsilon' \psi_j$ with $\psi_j \not\equiv 0$ and $\varepsilon' \in (0, \varepsilon^*]$, where $\varepsilon^* > 0$ is chosen suitably small so that Theorem 2.4 is valid. Then we can check that the corresponding φ_j^+ satisfies

$$\|\varphi_j^+\|_{L^2} = \|\alpha_j^+\|_{L^2} \geq \varepsilon' \|\psi_j\|_{L^2} - C^*(\varepsilon')^3$$

with some $C^* > 0$. Therefore φ_j^+ does not identically vanish if $\varepsilon' < \min\{\varepsilon^*, \sqrt{\|\psi_j\|_{L^2}/C^*}\}$.

A Proof of Lemma 4.1

In this appendix, we shall give the proof of Lemma 4.1 in the similar way as Section 2 of [2] with slight modifications. We first state the following useful lemma without proof, which is a special case of Lemma 2.1 of [2].

Lemma A.1. *We have*

$$\left\| \left[|\partial_x|^{1/2}, g \right] f \right\|_{L^2} + \left\| \left[|\partial_x|^{1/2} \mathcal{H}, g \right] f \right\|_{L^2} \leq C \|g\|_{W^{1,\infty}} \|f\|_{L^2}.$$

Proof of Lemma 4.1. As in the standard energy method, we compute

$$\frac{1}{2} \frac{d}{dt} \|Sv\|_{L^2}^2 = \operatorname{Im} \langle \mathcal{L}_m Sv, Sv \rangle_{L^2} = \operatorname{Im} \langle S \mathcal{L}_m v, Sv \rangle_{L^2} + \operatorname{Im} \langle [\mathcal{L}_m, S]v, Sv \rangle_{L^2}.$$

We also note that

$$[\mathcal{L}_m, S]v = -\frac{i}{|m|} \Phi S |\partial_x| v + Q,$$

where

$$Q = \frac{1}{2m} \Phi^2 Sv - \frac{i}{2|m|} (\partial_x \Phi) S \mathcal{H} v + \operatorname{sgn}(m) \left(\int_{-\infty}^x \partial_t \Phi(t, y) dy \right) S \mathcal{H} v.$$

Remark that $|\partial_x| = \mathcal{H} \partial_x = \partial_x \mathcal{H}$, $\mathcal{H}^2 = -1$, and that \mathcal{H} is L^2 -bounded. Now we set $w_k^{(l)} = \partial_x^l w_k$ for $l \in \mathbb{Z}_{\geq 0}$. Then, since

$$\begin{aligned} \partial_t \Phi &= 2\eta \sum_{l=0}^1 \sum_{k \in I_N} \operatorname{Im} \left\{ (i \partial_t w_k^{(l)}) \overline{w_k^{(l)}} \right\} \\ &= 2\eta \sum_{l=0}^1 \sum_{k \in I_N} \operatorname{Im} \left\{ \left(-\frac{1}{2\mu_k} \partial_x^2 w_k^{(l)} + \partial_x^l \mathcal{L}_{\mu_k} w_k \right) \overline{w_k^{(l)}} \right\} \\ &= 2\eta \sum_{l=0}^1 \sum_{k \in I_N} \operatorname{Im} \left\{ \partial_x \left(-\frac{1}{2\mu_k} (\partial_x w_k^{(l)}) \overline{w_k^{(l)}} \right) + \frac{1}{2\mu_k} |\partial_x w_k^{(l)}|^2 + (\partial_x^l \mathcal{L}_{\mu_k} w_k) \overline{w_k^{(l)}} \right\} \\ &= 2\eta \sum_{l=0}^1 \sum_{k \in I_N} \operatorname{Im} \left\{ \partial_x \left(-\frac{1}{2\mu_k} (\partial_x w_k^{(l)}) \overline{w_k^{(l)}} \right) + (\partial_x^l \mathcal{L}_{\mu_k} w_k) \overline{w_k^{(l)}} \right\}, \end{aligned}$$

we see that

$$\begin{aligned} \left| \int_{-\infty}^x \partial_t \Phi(t, y) dy \right| &= 2\eta \left| \sum_{l=0}^1 \sum_{k \in I_N} \operatorname{Im} \left\{ -\frac{1}{2\mu_k} (\partial_x w_k^{(l)}) \overline{w_k^{(l)}} + \int_{-\infty}^x (\partial_x^l \mathcal{L}_{\mu_k} w_k) \overline{w_k^{(l)}} dy \right\} \right| \\ &\leq C\eta \left(\|w\|_{W^{2,\infty}}^2 + \sum_{k \in I_N} \|\mathcal{L}_{\mu_k} w_k\|_{H^1} \|w_k\|_{H^1} \right). \end{aligned}$$

Therefore we obtain

$$\frac{d}{dt} \|Sv\|_{L^2}^2 + \frac{2}{|m|} \operatorname{Re} \langle \Phi S |\partial_x| v, Sv \rangle_{L^2} \leq 2 |\langle S \mathcal{L}_m v, Sv \rangle_{L^2}| + C B_1(t) \|Sv\|_{L^2}^2, \quad (\text{A.1})$$

where

$$B_1(t) = e^{C\|\Phi\|_{L^1}} \left(\|\Phi\|_{L^\infty}^2 + \|\partial_x \Phi\|_{L^\infty} + \eta \|w\|_{W^{2,\infty}}^2 + \eta \sum_{k \in I_N} \|\mathcal{L}_{\mu_k} w_k\|_{H^1} \|w_k\|_{H^1} \right).$$

Next we observe that

$$\begin{aligned} w_k^{(l)} S |\partial_x| v &= w_k^{(l)} S \partial_x \mathcal{H} v \\ &= \partial_x (w_k^{(l)} S \mathcal{H} v) + [w_k^{(l)} S, \partial_x] \mathcal{H} v \\ &= -|\partial_x|^{1/2} |\partial_x|^{1/2} \mathcal{H} w_k^{(l)} S \mathcal{H} v + [w_k^{(l)} S, \partial_x] \mathcal{H} v \\ &= |\partial_x|^{1/2} (w_k^{(l)} S |\partial_x|^{1/2} v) + [w_k^{(l)} S, \partial_x] \mathcal{H} v - |\partial_x|^{1/2} \left[|\partial_x|^{1/2} \mathcal{H}, w_k^{(l)} S \right] \mathcal{H} v, \end{aligned}$$

which leads to

$$\begin{aligned} \langle w_k^{(l)} S |\partial_x| v, w_k^{(l)} S v \rangle_{L^2} &= \langle w_k^{(l)} S |\partial_x|^{1/2} v, |\partial_x|^{1/2} (w_k^{(l)} S v) \rangle_{L^2} + \langle [w_k^{(l)} S, \partial_x] \mathcal{H} v, w_k^{(l)} S v \rangle_{L^2} \\ &\quad - \left\langle [|\partial_x|^{1/2} \mathcal{H}, w_k^{(l)} S] \mathcal{H} v, |\partial_x|^{1/2} (w_k^{(l)} S v) \right\rangle_{L^2} \\ &= \left\| w_k^{(l)} S |\partial_x|^{1/2} v \right\|_{L^2}^2 + X_{kl}, \end{aligned}$$

where

$$\begin{aligned} X_{kl} &= \left\langle w_k^{(l)} S |\partial_x|^{1/2} v, [|\partial_x|^{1/2}, w_k^{(l)} S] v \right\rangle_{L^2} + \left\langle [w_k^{(l)} S, \partial_x] \mathcal{H} v, w_k^{(l)} S v \right\rangle_{L^2} \\ &\quad - \left\langle [|\partial_x|^{1/2} \mathcal{H}, w_k^{(l)} S] \mathcal{H} v, w_k^{(l)} S |\partial_x|^{1/2} v \right\rangle_{L^2} - \left\langle [|\partial_x|^{1/2} \mathcal{H}, w_k^{(l)} S] \mathcal{H} v, [|\partial_x|^{1/2}, w_k^{(l)} S] v \right\rangle_{L^2}. \end{aligned}$$

By using Lemma A.1, we can see that all the commutators appearing in X_{kl} are L^2 -bounded and their operator norms are dominated by

$$B_2(t) = C e^{C\|\Phi\|_{L^1}} (\|w\|_{W^{2,\infty}} + \|w\|_{W^{1,\infty}} \|\Phi\|_{L^\infty}).$$

Hence we obtain

$$\begin{aligned}
& \left\| \sqrt{\Phi} S |\partial_x|^{1/2} v \right\|_{L^2}^2 - \operatorname{Re} \langle \Phi S |\partial_x| v, S v \rangle_{L^2} \\
&= \sum_{l=0}^1 \sum_{k \in I_N} \eta \operatorname{Re} \left(\left\| w_k^{(l)} S |\partial_x|^{1/2} v \right\|_{L^2}^2 - \langle w_k^{(l)} S |\partial_x| v, w_k^{(l)} S v \rangle_{L^2} \right) \\
&\leq \sum_{l=0}^1 \sum_{k \in I_N} \eta |X_{kl}| \\
&\leq C \eta B_2(t) \sum_{l=0}^1 \sum_{k \in I_N} \left\| w_k^{(l)} S |\partial_x|^{1/2} v \right\|_{L^2} \|v\|_{L^2} + C \eta B_2(t)^2 \|v\|_{L^2}^2 \\
&\leq \frac{1}{2} \left\| \sqrt{\Phi} S |\partial_x|^{1/2} v \right\|_{L^2}^2 + C \eta B_2(t)^2 \|v\|_{L^2}^2,
\end{aligned}$$

where we have used the Young inequality in the last line. Therefore,

$$\frac{2}{|m|} \operatorname{Re} \langle \Phi S |\partial_x| v, S v \rangle_{L^2} \geq \frac{1}{|m|} \left\| \sqrt{\Phi} S |\partial_x|^{1/2} v \right\|_{L^2}^2 - C \eta B_2(t)^2 \|v\|_{L^2}^2. \quad (\text{A.2})$$

From (A.1) and (A.2) it follows that

$$\frac{d}{dt} \|Sv\|_{L^2}^2 + \frac{1}{|m|} \left\| \sqrt{\Phi} S |\partial_x|^{1/2} v \right\|_{L^2}^2 \leq 2 |\langle S \mathcal{L}_m v, S v \rangle_{L^2}| + C (B_1(t) + \eta B_2(t)^2) \|Sv\|_{L^2}^2.$$

Finally, by using $\|\Phi\|_{L^1} \leq C \eta \|w\|_{H^1}^2$, $\|\Phi\|_{L^\infty} \leq C \eta \|w\|_{W^{1,\infty}}^2$ and $\|\partial_x \Phi\|_{L^\infty} \leq C \eta \|w\|_{W^{2,\infty}}^2$, we have

$$\begin{aligned}
B_1(t) + \eta B_2(t)^2 &\leq C e^{C \eta \|w\|_{H^1}^2} \left(\eta^2 \|w\|_{W^{1,\infty}}^4 + \eta \|w\|_{W^{2,\infty}}^2 + \eta \sum_{k \in I_N} \|\mathcal{L}_{\mu_k} w_k\|_{H^1} \|w_k\|_{H^1} \right) \\
&\quad + C \eta e^{C \eta \|w\|_{H^1}^2} (\|w\|_{W^{2,\infty}}^2 + C \eta^2 \|w\|_{W^{1,\infty}}^6) \\
&\leq C e^{C \eta \|w\|_{H^1}^2} \left(\eta \|w\|_{W^{2,\infty}}^2 + \eta^3 \|w\|_{W^{1,\infty}}^6 + \eta \sum_{k \in I_N} \|\mathcal{L}_{\mu_k} w_k\|_{H^1} \|w_k\|_{H^1} \right),
\end{aligned}$$

which yields the desired conclusion. \square

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